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Multiple Pole Rational-Function Approximations for Unsteady Aerodynamics

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Nomenclature

$[A]$ = coefficient matrices
b = reference length
b_n = poles (lag-parameters)
$[Q]$ = unsteady aerodynamic transfer-function matrix
Q_{ij} = element (i, j) of $[Q]$
s = Laplace variable
U = freestream velocity

Introduction

FOR a general aeroservoelastic analysis, the equations of motion are desired in a linear, time-invariant, state-space form. This necessitates the representation of the unsteady aerodynamic transfer function matrix, for a general motion in the Laplace domain, by a rational-function approximation (RFA) for each term of the matrix. Since the unsteady aerodynamic transfer-function matrix $[Q(s)]$ is analytic for a causal, stable, and linear system, it can be directly deduced from the frequency domain data through a process of analytic continuation, which involves a least squares curve-fit.

Several approaches have been used to determine the poles (lag-parameters) of $[Q(s)]$ by a nonlinear optimization process. Dunn,[1] Karpel,[2] and Peterson and Crawley,[3] used gradient-based optimization schemes, whereas, Refs. 4–6, and 9 employed Simplex nongradient techniques. Peterson and Crawley[3] observed the phenomenon of repeated poles in approximating for the Theodorsen function. However, the repeated lag-states mistakenly indicated that the same fit-accuracy can be achieved by reducing the number of lag-states. Eversman and Tewari[5] encountered the repeated values of lag-parameters frequently in a nongradient optimized RFA, and correctly identified the phenomenon to indicate the need for a new multiple-pole approximation in the Laplace domain. Reference 5 showed that while the conventional approximation of simple poles produces an ill-conditioned eigenvalue problem for the state-space model when the poles are close to one another, the new multiple-pole RFA accounts for such cases consistently. Additionally, the use of multiple-poles resulted in a large reduction in the optimization cost, while preserving the fit-accuracy and the total number of aerodynamic states when compared to the conventional approximation. Eversman and Tewari[6] also presented improved and consistent RFA for the Theodorsen function by using the multiple-pole approximation. Tewari,[9] in a Ph.D. dissertation, showed that the multiple-pole RFA is needed not only in the subsonic regime, but also for supersonic speeds. References 5 and 9 arrived at the multiple-pole RFA through numerical considerations. The present work examines the multiple-pole RFA from a mathematical standpoint and validates its need by concluding that multiple-pole RFA is dictated in the function space by the constrained optimization theory.

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Numerical Need for Multiple-Pole RFA

A simple-pole, least-squares RFA for the unsteady aerodynamic transfer-function matrix can be expressed as

$$[Q(s)] = [A_0] + [A_1]s(b/U) + [A_2]s^2(b/U)^2 + (U/b)\sum_{n=1}^{nL}\frac{[A_{(n+2)}]}{s + (b/U)b_n} \quad (1)$$

Reference 5 showed that the optimized values of two or more lag parameters b_n frequently tend to be close to one another for a subsonic numerical test case. It was also shown that when repeated poles occur, numerical considerations point toward the need for a multiple-pole RFA, given by

$$[Q(s)] = [A_0] + [A_1]s(b/U) + [A_2]s^2(b/U)^2 + (U/b) \cdot\sum_{n=1}^{N_1}\frac{[A_{(n+2)}]}{s + (U/b)b_n} + (U/b)^2\sum_{n=N_1+1}^{N_2}\frac{[A_{(n+2)}]}{[s + (U/b)b_n]^2} + \ldots \quad (2)$$

where N_1 is the total number of poles, $(N_2 - N_1)$ the number of poles repeated twice or more times, etc. Such RFA avoid the ill-conditioned eigenvalue problem produced by having repeated poles in Eq. (1).

While Ref. 5 studied the RFA for the subsonic case, it was subsequently discovered[9] that repeated poles are equally prevalent in the supersonic regime, and that a multiple-pole RFA, given by Eq. (2), produces a consistent and efficient approximation for supersonic speeds. In the test-case considered, the same planform geometry was used as in Ref. 5, but the wing was stiffened in order to have the flutter-speed in the supersonic regime. The supersonic "doublet-point" method[7–9] was used to generate the frequency domain data at the following set of reduced frequencies:

0.0, 0.05, 0.1, 0.125, 0.175

0.2, 0.3, 0.5, 0.7, 0.9, 1.2

Only the first six structural modes were retained. Table 1 presents the optimum pole values for supersonic Mach numbers. As in Ref. 5 for subsonic Mach numbers, it is noted

Table 1 Optimum lag-parameter values for supersonic Mach numbers

M	b_1	b_1, b_2	b_1, b_2 b_3	b_1, b_2 b_3, b_4
1.05	0.35911	0.85623, 0.85627	0.45076, 0.45202 0.45175	0.44029, 0.43670 0.44041, 0.00022
1.1	0.34817	0.09694, 0.95183	0.29519, 0.29429 0.29474	0.21630, 0.22926 0.22435, 0.22786
1.2	0.61702	0.06371, 0.87254	0.10342, 0.42278 0.42276	0.11742, 0.11751 1.55287, 1.55614
1.3	0.31952	0.24700, 1.23706	0.17274, 0.17274 1.02893	0.35536, 0.34634 0.33700, 0.34960
1.4	0.34735	0.34871, 1.80238	0.83560, 0.77938 0.77943	1.82796, 0.16268 0.16165, 0.16041
1.5	0.45369	0.96795, 0.96795	0.00108, 0.28711 0.28719	1.01826, 1.27046 1.19240, 1.17105
1.6	0.39346	0.73422, 0.73489	0.17320, 0.17401 0.17315	0.96374, 0.90355 0.90890, 0.76043

Table 2 New lag-parameters for $M = 1.05$

Number of lag-states	Lag-parameters, Eq. (1)	Lag-parameters, Eq. (2)
2	0.856229 0.856272 (Fit-error = 16.901)	0.856273 (Double-pole) (Fit-error = 16.903)
3	0.450756 0.452024 0.451749 (Fit-error = 5.212)	0.451269 (Triple-pole) (Fit error = 5.498)
4	0.44029 0.43670 0.44041 0.00022 (Fit-error = 4.344)	0.439976 (Triple-pole) 0.000186 (Simple-pole) (Fit-error = 4.365)

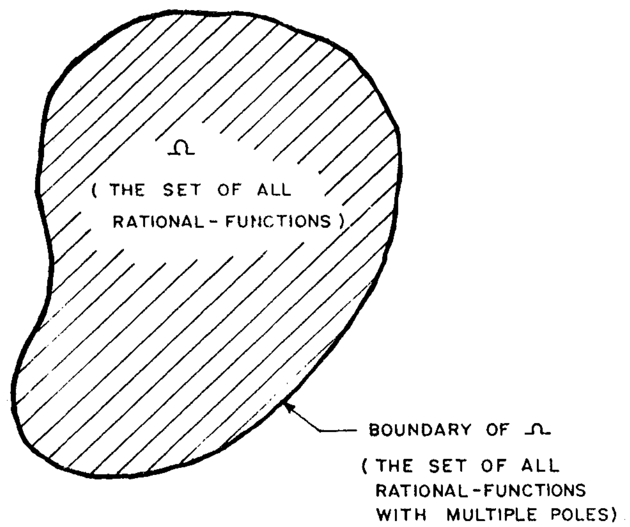

Fig. 1 Feasible set for nonlinear optimization.

that the new multiple-pole RFA effectively replaces the repeated pole cases for supersonic speeds, as seen in Table 2 for $M = 1.05$, which is typical. From computational considerations, a multiple-pole RFA is considerably more efficient when compared to a conventional simple-pole RFA of equivalent accuracy, since the number of nonlinear parameters (poles) in the optimization process are reduced. Therefore, even though the optimal RFA may not have repeated poles for all Mach numbers (Table 1), it can be replaced by a multiple-pole RFA without sacrificing the curve-fit accuracy or increasing the number of aerodynamic states.[5,9]

Analytical Look at the Need for Multiple Poles

It is not surprising from the mathematical viewpoint that the need for a multiple-pole RFA should frequently arise. Let Ω be the set of all rational-functions with a fixed order of poles. The rational-function with multiple-poles occur on the boundary of Ω (Fig. 1). The nonlinear optimization problem for the determination of optimum poles can be considered as the minimization of the least-squares fit-error of the RFA with the frequency-domain data, subject to the constraint $Q_{ij} \in \Omega$. This type of constraint is referred to as a set-constraint.[10] Since the constrained optimization process often yields solutions on the boundary of the feasible set,[10] Ω, which is defined as the set of all multiple-pole rational-functions, it follows that the optimal rational-functions would frequently have multiple poles.

Conclusion

The phenomenon of repeated poles (lag-parameters) in a rational-function approximation for the unsteady aerodynamic transfer-function matrix is frequently encountered not only at subsonic but also at supersonic speeds. As with the subsonic case, a multiple-pole approximation accurately, efficiently, and consistently replaces the conventional, simple-pole approximation in the supersonic regime. When considered from the mathematical perspective of constrained, nonlinear function minimization theory, the observed need for multiple-pole RFA is easily explained. The multiple-pole rational-functions form a boundary for the set of all rational-functions (with the same order of poles), which is also the feasible set for the optimization problem in the function space.

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Neutrally Reinforced Holes in Symmetrically Laminated Plates

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Introduction

SINCE laminated composite plates with openings are widely used structural elements in many engineering applications, the analysis of such structures has been done by numerous researchers. The stress and strain state around the openings have been presented for a variety of problems. A large number of problem solutions are presented in the monographs by Lekhnitskii[1,2] and Savin.[3] In the formulation of these problems, the material constitutive law, the geometry, loading, and boundary conditions are assumed to be given and the stress and strain state are computed.

In a particular class of structural optimization, the geometry of the body is obtained as a solution to a particular problem, for known states of stress, strain, boundary condition and appropriate constraints (sometimes called inverse problems for some specific cases). In the latter category, either stress concentrations are reduced (optimized) by determining the shape of the openings (harmonic holes), or stress states in the cut structures with reinforcement are maintained unchanged to that of the uncut structures (neutral holes).

All previous solutions,[3-5] for neutral holes have been obtained for elastic, isotropic sheets under planar states of stress. In a previous paper,[6] the methodology outlined in Ref. 1 was extended to symmetrically laminated composite plates which are under planar loading. In the present work, the case of a symmetrically laminated plate subjected to pure bending moments is considered. It must be noted that the notion of a neutral hole in the context of plate flexure assumes a new meaning. Here, the purpose is to introduce a reinforced cutout into a plate that will maintain the same moment and curvature distributions as of the uncut plate throughout.

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Formulation

Laminated Plate Under Bending

Let a Cartesian coordinate system be located in the midplane of the laminate, about which the plate is laminated symmetrically, such that the x-y plane coincides with the lamination plane and the z axis is perpendicular to that plane. Under "classical laminate theory" assumptions, the lateral force and moment equilibrium equations are the following[8]:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0$$
$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x \qquad (1)$$
$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = Q_y$$

Here, M_x, M_y, and M_{xy} are the moment resultants (moment per unit length), Q_x, Q_y are the shear forces per unit length, and q is the lateral surface force per unit area acting on the laminate. From Eq. (1) the following can be deduced in the absence of a lateral force q. In this way, the equilibrium is represented with one equation:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} = 0 \qquad (2)$$

The reinforcing member is assumed as a beam element which has flexural as well as twisting rigidity. With reference to Figs. 1 and 2 the out-of-plane force and moment equilib-

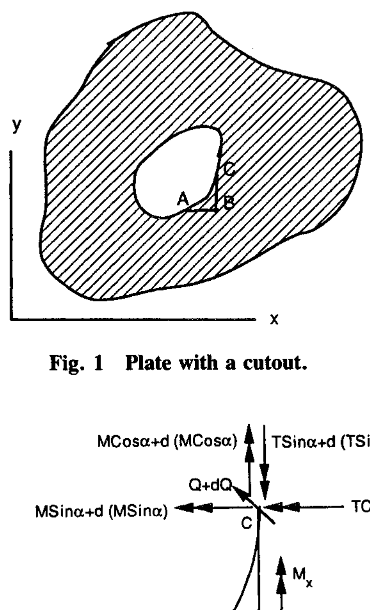

Fig. 1 Plate with a cutout.

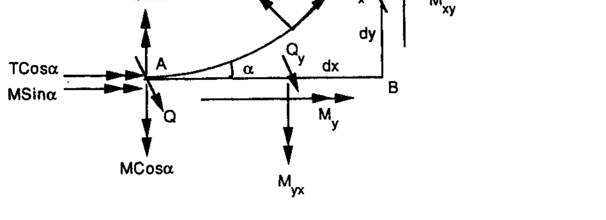

Fig. 2 Moments and out-of-plane forces acting on the element ABC.